Local currents for a deformed algebra of quantum mechanics with a fundamental length scale

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Abstract
We explore some explicit representations of a certain stable deformed algebra of quantum mechanics, considered by R Vilela Mendes, having a fundamental length scale. The relation of the irreducible representations of the deformed algebra to those of the (limiting) Heisenberg algebra is discussed, and we construct the generalized harmonic oscillator Hamiltonian in this framework. To obtain local currents for this algebra, we extend the usual nonrelativistic local current algebra of vector fields and the corresponding group of diffeomorphisms, modelling the quantum configuration space as a commutative spatial manifold with one additional dimension.

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1. Introduction

The possibility of experimentally observing features of quantum gravity at small length scales has heightened interest in the study of space-time noncommutativity [1, 2]—e.g., through the mathematics of noncommutative geometry [3], and/or as a feature of string theories [4–6]. The characteristic length scale ℓ at which nonclassical features of gravity should emerge may be the Planck length $\ell_P = \sqrt{\hbar G/c^3} \sim 1.6 \times 10^{-35}$ m, or it may be significantly larger [7].

One way to introduce such noncommutativity is algebraic. A few years ago, Vilela Mendes [8] argued again for consideration of the combined Heisenberg and Poincaré Lie algebras as a kinematical algebra for relativistic quantum mechanics. This structure is 'unstable', but allows a parameterized family of nontrivial deformations that are 'stable'—in the sense that all the Lie algebras in an open neighbourhood in the space of structure constants are mutually isomorphic [9–12]. The nontrivial second cohomology of the original Lie algebra is a necessary condition for it to be deformable [13]. The proposed stable algebra for relativistic quantum

mechanics is a deformation by two parameters ℓ and R, which are fundamental lengths. Taking $\ell \to 0$ and $R \to \infty$ leads to recovery of the original Lie algebra. Vilela Mendes argues on fundamental grounds for describing the physical world by means of a stable Lie algebra, so that small changes in physical constants do not fundamentally alter the structure. We take no position here on this question, but explore some interesting consequences of the Vilela Mendes approach. A recent, beautiful paper of Chryssomalakos and Okon describes and discusses the full set of possible stable deformations of the Heisenberg–Poincaré algebra, with explanation of the relevant cohomology theory and detailed references [14].

The present paper is motivated by the problem of defining an equal-time, local current algebra compatible with the nonrelativistic quantum kinematics that follows from Vilela Mendes' proposal. Like him we consider the case where $R \to \infty$, but $\ell \neq 0$; then the space-time coordinate operators no longer commute. We clarify the relation of the irreducible representations of a deformed subalgebra to those of the limiting Heisenberg algebra, concentrating on the case of one space dimension (although our considerations generalize straightforwardly to higher dimensions). The limit procedure here goes back to a 1970 scheme of Barut and Bohm for reduction of certain representations of $SO(4, 2)$ [15]. But our construction of the generalized kinetic energy and harmonic oscillator Hamiltonians in this framework leads to an answer different from that suggested by Vilela Mendes.

One way of obtaining local currents for the deformed algebra is to extend the usual nonrelativistic local current algebra (LCA) of scalar functions and vector fields, and the corresponding infinite-dimensional groups of scalar functions and diffeomorphisms. In doing this we make use of an abstract single-particle configuration space, which is a commutative spatial manifold having one dimension more than the configuration space for the limiting situation with $\ell \to 0$. Thus, the deformed (1+1)-dimensional theory entails self-adjoint representations of an infinite-dimensional Lie algebra of nonrelativistic, local currents for a (2+1)-dimensional space-time (LCA2). To be able to recover the usual current algebra (LCA1) in the limit $\ell \to 0$, one may introduce a semidirect sum of LCA2 with the algebra of vector fields of the line. The local operators then act in a direct integral of irreducible representations of the global, finite-dimensional deformed Lie algebra. This seems to open interesting new possibilities, which we discuss briefly.

The paper is organized as follows. In section 2, we present the necessary background— the (ℓ, R)-deformed Lie algebra of relativistic quantum mechanics, the nonrelativistic local current algebra and its relation to the Heisenberg algebra, and the desirable properties for local currents in relation to the deformed algebra with $\ell \neq 0$. In section 3, we review several different, but unitarily equivalent, self-adjoint representations of the ℓ-deformed Heisenberg algebra for the case of one spatial dimension—which is isomorphic to the Lie algebra of the group of rigid motions of the plane. This subalgebra for the one-dimensional problem serves as a useful laboratory. We write explicitly the unitary operators intertwining the representations. This permits clarification of how an irreducible representation of the usual Heisenberg algebra can be recovered in the $\ell \to 0$ limit. In section 4, we discuss the kinetic energy and harmonic oscillator Hamiltonians. In section 5, we consider first the problems associated with currents localized with respect to the spectrum of the deformed position operator. Then we develop and discuss the extended nonrelativistic local current algebra and diffeomorphism group.

2. Background

Introducing the 4-vectors q_μ and p_ν, $\mu, \nu = 0, 1, 2, 3$, and the Lorentz generators $M_{\mu\nu}$, one combines the canonical brackets

$$[p_\mu, q_\nu] = i\hbar\eta_{\mu\nu}\mathcal{J}, \qquad [q_\mu, q_\nu] = [p_\mu, p_\nu] = [q_\mu, \mathcal{J}] = [p_\mu, \mathcal{J}] = 0, \tag{1}$$

with the Lorentz brackets

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(M_{\mu\sigma}\eta_{\nu\rho} + M_{\nu\rho}\eta_{\mu\sigma} - M_{\nu\sigma}\eta_{\mu\rho} - M_{\mu\rho}\eta_{\nu\sigma}), \tag{2}$$

by means of the additional brackets

$$
\begin{aligned}
[M_{\mu\nu}, p_\lambda] &= i(p_\mu\eta_{\nu\lambda} - p_\nu\eta_{\mu\lambda}), \\
[M_{\mu\nu}, q_\lambda] &= i(q_\mu\eta_{\nu\lambda} - q_\nu\eta_{\mu\lambda}), \\
[M_{\mu\nu}, \mathcal{J}] &= 0,
\end{aligned}
\tag{3}
$$

where $\eta_{\mu\nu} = \mathrm{diag}[1, -1, -1, -1]$ in units with $c = 1$. To describe the quantum kinematics of a particle, one typically represents a *subalgebra* of this Lie algebra by self-adjoint operators in a Hilbert space.

While the Lie algebras of equations (1) and (2) are separately stable, the combined Lie algebra of equations (1)–(3) is not, so we consider a stable deformation.

2.1. Deformed Lie algebras for quantum mechanics

The relevant deformation is labelled by fundamental lengths R and ℓ, and satisfies brackets where equations (2) and (3) are unchanged, but equations (1) are replaced by

$$[p_\mu, q_\nu] = i\hbar\eta_{\mu\nu}\mathcal{J} \qquad [q_\mu, q_\nu] = -i\varepsilon\ell^2 M_{\mu\nu}, \qquad [p_\mu, p_\nu] = -i\frac{\varepsilon'\hbar^2}{R^2}M_{\mu\nu},$$

$$[q_\mu, \mathcal{J}] = i\varepsilon\frac{\ell^2}{\hbar}p_\mu, \qquad [p_\mu, \mathcal{J}] = -i\frac{\varepsilon'\hbar}{R^2}q_\mu, \tag{4}$$

where ε and ε' are ± 1. Evidently as $\ell \to 0$ and $R \to \infty$, we recover equations (1)–(3).

This Lie algebra is isomorphic to the Lie algebra of the orthogonal group in six dimensions, with metric $\eta_{ab} = \mathrm{diag}[1, -1, -1, -1, \varepsilon', \varepsilon]$. Evidently, in a self-adjoint representation, q_μ no longer commute with each other. Their interpretation as space-time coordinate operators in such a representation may be questioned [14], but if we maintain this interpretation, Heisenberg-like uncertainty relations for these coordinates suggest that space-time becomes 'fuzzy' to order ℓ [16].

We remark that in other specific models, the noncommutativity of the coordinate operators is different. For example, in the case of a charged particle moving in a plane perpendicular to a magnetic field of magnitude B, in the limit as the mass $\mathfrak{m} \to 0$ we expect, for $j, k = 1, 2$,

$$[q_j, q_k] = i\theta_{jk}\mathcal{J}, \qquad [q_j, \mathcal{J}] = 0, \tag{5}$$

where θ is a constant antisymmetric matrix inversely proportional to B. A similar bracket occurs for a bosonic string when there is a background, constant Neveu–Schwarz 2-form in the world volume of a D-brane [17, 18].

As the parameter ℓ is relevant locally, we shall follow Vilela Mendes in focusing on the algebra obtained by taking $R \to \infty$. Then the brackets involving R in equations (4) become zero. We now want to concentrate on self-adjoint representations of the Heisenberg-like subalgebra, with $j, k = 1, 2, 3$, and $\varepsilon = -1$, given by the spatial components of equations (2) and (3), together with the brackets

$$
\begin{aligned}
[q_j, q_k] &= i\ell^2 M_{jk}, \qquad [q_j, p_k] = i\delta_{jk}\hbar\mathcal{J}, \\
[q_j, \mathcal{J}] &= -i\frac{\ell^2}{\hbar}p_j, \qquad [p_j, p_k] = [p_j, \mathcal{J}] = 0.
\end{aligned}
\tag{6}
$$

The Lie algebra of equations (2), (3) and (6) represents the *global* symmetry of the deformed quantum theory. We desire, however, to incorporate a description of *local* symmetry, for which we turn to the local current algebra.

2.2. Equal-time local current algebra

In standard, nonrelativistic quantum theory, the second-quantized field $\widehat{\psi}(\mathbf{x}, t)$ and its adjoint $\widehat{\psi}^*(\mathbf{x}, t)$, for $\mathbf{x} \in \mathbb{R}^d$, are operator-valued distributions in the Fock space. These satisfy the equal-time canonical commutation $(-)$ or anticommutation $(+)$ relations,

$$[\widehat{\psi}(\mathbf{x}), \widehat{\psi}^*(\mathbf{y})]_{\pm(\text{fixed}\,t)} = \delta(\mathbf{x} - \mathbf{y}), \tag{7}$$

the argument t is henceforth suppressed. Then the local, fixed-time mass density $\rho(\mathbf{x})$ and momentum density $J(\mathbf{x})$ are operator-valued distributions, defined formally by

$$\rho(\mathbf{x}) = \mathfrak{m}\widehat{\psi}^*(\mathbf{x})\widehat{\psi}(\mathbf{x}), \qquad \mathbf{J}(\mathbf{x}) = \frac{\hbar}{2\mathrm{i}}\{\widehat{\psi}^*(\mathbf{x})\nabla\widehat{\psi}(\mathbf{x}) - [\nabla\widehat{\psi}^*(\mathbf{x})]\widehat{\psi}(\mathbf{x})\}. \tag{8}$$

These obey a certain singular Lie algebra, which is independent of whether the original field is bosonic or fermionic [19]. Define

$$\rho(f) = \int \rho(\mathbf{x}) f(\mathbf{x})\, \mathrm{d}\mathbf{x}, \qquad J(\mathbf{g}) = \int \sum_{k=1}^{d} J_k(\mathbf{x}) g_k(\mathbf{x})\, \mathrm{d}\mathbf{x}, \tag{9}$$

where f and the components g_k of $\mathbf{g}$ are compactly-supported C^∞ test functions on $\mathbb{R}^d$. Then one obtains the local current algebra [20]

$$[\rho(f_1), \rho(f_2)] = 0, \qquad [\rho(f), J(\mathbf{g})] = \mathrm{i}\hbar\rho(\mathbf{g} \cdot \nabla f),$$
$$[J(\mathbf{g}_1), J(\mathbf{g}_2)] = -\mathrm{i}\hbar J([\mathbf{g}_1, \mathbf{g}_2]), \tag{10}$$

where $[\mathbf{g}_1, \mathbf{g}_2] = \mathbf{g}_1 \cdot \nabla\mathbf{g}_2 - \mathbf{g}_2 \cdot \nabla\mathbf{g}_1$ is the usual Lie bracket of vector fields.

In the one-particle Hilbert space $L^2_{\mathrm{dx}}(\mathbb{R}^d)$, we have the self-adjoint representation

$$\rho(f)\Psi(\mathbf{x}) = \mathfrak{m}f(\mathbf{x})\Psi(\mathbf{x}), \qquad J(\mathbf{g})\Psi(\mathbf{x}) = \frac{\hbar}{2\mathrm{i}}\{\mathbf{g}(\mathbf{x}) \cdot \nabla\Psi(\mathbf{x}) + \nabla \cdot [\mathbf{g}(\mathbf{x})\Psi(\mathbf{x})]\}, \tag{11}$$

where $\mathfrak{m}$ is the particle mass. Now as the test function $f(\mathbf{x})$ approaches an indicator function $\chi_B(\mathbf{x})$ for a Borel set $B \subseteq \mathbb{R}^d$, the expectation value $(\Psi, \rho(f)\Psi)$ with respect to the single-particle wavefunction Ψ approximates $\mathfrak{m} \int \chi_B(\mathbf{x})|\psi(\mathbf{x})|^2\mathrm{d}\mathbf{x}$, which is the mass times the usual probability for finding the particle in the region B. If $f(\mathbf{x})$ approaches $\delta(\mathbf{x} - \mathbf{x}_0)$ for a fixed point $\mathbf{x}_0 \in \mathbb{R}^d$, then $(\Psi, \rho(f)\Psi)$ approaches $\mathfrak{m}|\Psi(\mathbf{x}_0)|^2$. We also see how the Heisenberg algebra is recovered—if $f(\mathbf{x})$ approximates the coordinate function x_j, then $\rho(f)$ approximates the moment operator $\mathfrak{m}q_j$ acting in $L^2_{\mathrm{dx}}(\mathbb{R}^d)$ via multiplication by $\mathfrak{m}x_j$. Similarly, if $\mathbf{g}(\mathbf{x})$ is taken to approximate a constant vector field in the j-direction, so that (let us say) $g_j(\mathbf{x}) \sim 1$ with $g_k(\mathbf{x}) = 0$ for $k \neq j$, then $J(\mathbf{g}) \sim -\mathrm{i}\hbar\partial/\partial x_j$, which is the action of the momentum operator p_j in $L^2_{\mathrm{dx}}(\mathbb{R}^d)$.

In short, the LCA in the one-particle representation, with suitable (global) choices of test functions, allows recovery of the usual quantum-mechanical representation of the finite-dimensional subalgebra of equations (1) having spatial indices. Likewise, generators of spatial rotations may be recovered—e.g., in the one-particle representation in three space dimensions, the operator for the orbital angular momentum about the x_3-axis is approximated by choosing $g_1(\mathbf{x}) = -x_2, g_2(\mathbf{x}) = x_1$, and $g_3(\mathbf{x}) = 0$ inside a large compact region $|\mathbf{x}| \leqslant R$; outside this region, $\mathbf{g}(\mathbf{x})$ falls smoothly to 0. Then $J(\mathbf{g})$ approximates the operator $\hbar M_{12} = L_3 = (\mathbf{q} \times \mathbf{p}) \cdot \mathbf{e}_3$ acting in $L^2_{\mathrm{dx}}(\mathbb{R}^3)$, where $\mathbf{e}_3$ is the unit vector in the x_3-direction.

The study of inequivalent, self-adjoint representations of this infinite-dimensional algebra has turned out to be a powerful method for classifying, and in some cases predicting, kinematical possibilities for quantum systems. These possibilities include the usual N-particle representations, $N = 1, 2, 3, \ldots$, satisfying bosonic or fermionic statistics for $N \geqslant 2$ in more than one space dimension. They also include particle systems obeying anyonic statistics

in the two-dimensional space [21–25] or other exotic statistics, particles with spin [26, 27], composite systems having dipole or higher multipole moments [28], and infinite particle or extended systems having infinite-dimensional configuration spaces [29–31].

This is our motivation for investigating the possibilities for defining local currents appropriate to the deformed Lie algebra of equations (6). The goal is to obtain an infinite-dimensional, local Lie algebra that is a deformation or extension of the LCA of equations (10), which, with suitable choices of test functions in a one-particle representation, allows recovery of a standard representation of equations (6). Then the unitarily inequivalent representations of the deformed LCA should describe kinematical possibilities for a nonrelativistic version of the deformed quantum theory. Thus in section 3, we shall discuss some different ways of writing standard representations of equations (10).

Note that we may introduce an operator-valued distribution $Q(f, \mathbf{g})$ acting in $L^2_{d\mathbf{x}}(\mathbb{R}^d)$, defined by

$$Q(f, \mathbf{g})\Psi = f(\mathbf{x})\Psi + \frac{1}{2i}\{\mathbf{g}(\mathbf{x}) \cdot \nabla\Psi(\mathbf{x}) + \nabla \cdot [\mathbf{g}(\mathbf{x})\Psi(\mathbf{x})]\}. \tag{12}$$

Then Q is a self-adjoint representation of the natural semidirect sum of the commutative Lie algebra of compactly-supported, real-valued C^∞ functions f on $\mathbb{R}^d$, with the Lie algebra of vector fields $\mathbf{g}$ on $\mathbb{R}^d$; namely,

$$[(f_1, \mathbf{g}_1), (f_2, \mathbf{g}_2)] = (\mathbf{g}_2 \cdot \nabla f_1 - \mathbf{g}_1 \cdot \nabla f_2, -[\mathbf{g}_1, \mathbf{g}_2]). \tag{13}$$

The physical constants $\mathfrak{m}$ and $\hbar$ do not enter (12), but equations (11) follow from it when we set $\rho(f) = \mathfrak{m}Q(f, 0)$ and $J(\mathbf{g}) = \hbar Q(0, \mathbf{g})$.

The group that is associated with equation (13) is the natural semidirect product $\mathcal{D}(\mathbb{R}^d) \times \mathrm{Diff}^c(\mathbb{R}^d)$, where $\mathcal{D}(\mathbb{R}^d)$ is the group of compactly supported, real-valued C^∞ functions on $\mathbb{R}^d$ under pointwise addition, and $\mathrm{Diff}^c(\mathbb{R}^d)$ is the group of compactly supported diffeomorphisms of $\mathbb{R}^d$ under composition [31]. These groups are endowed with the topology of uniform convergence in all derivatives on compact sets. The group law, for $f_1, f_2 \in \mathcal{D}(\mathbb{R}^d)$ and $\phi_1, \phi_2 \in \mathrm{Diff}^c(\mathbb{R}^d)$, is

$$(f_1, \phi_1)(f_2, \phi_2) = (f_1 + f_2 \circ \phi_1, \phi_2 \circ \phi_1), \tag{14}$$

where $\circ$ denotes composition.

Given the compactly-supported, C^∞ vector field $\mathbf{g}$ on $\mathbb{R}^d$, there exists a unique, one-parameter group of C^∞ diffeomorphisms $\phi^{\mathbf{g}}_a(\mathbf{x})$, $a \in \mathbb{R}$, such that

$$\frac{\partial \phi^{\mathbf{g}}_a(\mathbf{x})}{\partial a} = \mathbf{g}(\phi^{\mathbf{g}}_a(\mathbf{x})) \tag{15}$$

with the initial condition $\phi^{\mathbf{g}}_{a=0}(\mathbf{x}) \equiv \mathbf{x}$. In a continuous, unitary representation $U(f)V(\phi)$ of $\mathcal{D}(\mathbb{R}^d) \times \mathrm{Diff}^c(\mathbb{R}^d)$, the local currents are the self-adjoint generators of one-parameter unitary subgroups; so that

$$\begin{aligned} U(f) &= \exp[(i/\mathfrak{m})\rho(f)] = \exp[iQ(f, 0)], \\ V(\phi^{\mathbf{g}}_s) &= \exp[(is/\hbar)J(\mathbf{g})] = \exp[iQ(0, \mathbf{g})]. \end{aligned} \tag{16}$$

The method of induced representations, and other techniques of unitary group representation, have been extensively used in the study of the local current algebra.

3. Some unitarily equivalent representations

Let us now turn to some representations of the spatial components of equations (2)–(3), taken together with equations (6). In d space dimensions, there is a natural representation by

derivations over the (commutative) manifold $\mathbb{R}^{d+1}$, with global coordinates $(x_1, x_2, \ldots, x_d, w)$. Labelling this representation with the superscript (1), it is given by [8]

$$q_j^{(1)} = \mathrm{i}\ell\left(w\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial w}\right), \qquad M_{jk}^{(1)} = -\mathrm{i}\left(x_j\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial x_j}\right),$$
$$\mathcal{J}^{(1)} = -\mathrm{i}\ell\frac{\partial}{\partial w}, \qquad\qquad\qquad p_j^{(1)} = -\mathrm{i}\hbar\frac{\partial}{\partial x_j}. \tag{17}$$

The algebra is represented here by the generators of rigid motions of $\mathbb{R}^{d+1}$, i.e. the $(d + 1)$-dimensional Euclidean group E_{d+1} which is a semidirect product of the translations and rotation groups. Note that the coordinate w is like a hidden dimension in Kaluza–Klein theory [32]; its presence here illustrates the idea that sometimes noncommutative structures can serve as alternatives to hidden dimensions. The Hilbert space on which the differential operators of equations (17) act as self-adjoint operators is the space $\mathcal{H} = L^2_{\mathrm{d}\mathbf{x}\,\mathrm{d}w}$, consisting of complex-valued functions Φ on $\mathbb{R}^{d+1}$ that are square-integrable with respect to the Lebesgue measure $\left(\prod_{j=1}^d \mathrm{d}x_j\right)\mathrm{d}w$.

But the representation of equations (17) has the drawback that when we take the limit as $\ell \to 0$, we do not recover the usual representation of the Heisenberg algebra without deformation; rather $q_j^{(1)}$ and $\mathcal{J}^{(1)}$ both tend formally to 0. We therefore consider an alternative, denoted with the superscript (2), obtained by introducing a unitary multiplication operator U_ℓ in the Hilbert space $\mathcal{H}$. Defining $U_\ell\Phi(\mathbf{x}, w) = \exp[-\mathrm{i}w/\ell]\Phi(\mathbf{x}, w)$, we set

$$\left(q_j^{(2)}, M_{jk}^{(2)}, \mathcal{J}^{(2)}, p_j^{(2)}\right) = U_\ell\left(q_j^{(1)}, M_{jk}^{(1)}, \mathcal{J}^{(1)}, p_j^{(1)}\right)U_\ell^{-1}, \tag{18}$$

then we have

$$q_j^{(2)} = x_j + \mathrm{i}\ell\left(w\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial w}\right), \qquad M_{jk}^{(2)} = -\mathrm{i}\left(x_j\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial x_j}\right),$$
$$\mathcal{J}^{(2)} = I - \mathrm{i}\ell\frac{\partial}{\partial w}, \qquad\qquad p_j^{(2)} = -\mathrm{i}\hbar\frac{\partial}{\partial x_j}. \tag{19}$$

Now the operators smoothly go over to the standard Heisenberg representation as $\ell \to 0$. This representation is also given in [8].

Let us introduce two corresponding unitarily equivalent representations obtained by the Fourier transformation. As usual, we have the unitary operator

$$\mathcal{F} : \mathcal{H} = L^2_{\mathrm{d}\mathbf{x}\,\mathrm{d}w} \to \widehat{\mathcal{H}} = L^2_{\mathrm{d}\mathbf{k}_x\,\mathrm{d}k_w},$$

given by

$$[\mathcal{F}\Phi](k_1, \ldots, k_d, k_w) = \widehat{\Phi}(k_1, \ldots, k_d, k_w) = \left(\frac{1}{\sqrt{2\pi}}\right)^{d+1}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathrm{d}x_1\cdots\mathrm{d}x_d\,\mathrm{d}w$$
$$\cdot\,\Phi(\mathbf{x}, w)\exp[-\mathrm{i}\mathbf{x}\cdot\mathbf{k}_x - \mathrm{i}k_w w], \tag{20}$$

where $\mathbf{k}_x = (k_1, \ldots, k_d)$ and k_w are the Fourier conjugate variables to $\mathbf{x}$ and w, respectively. Then setting

$$\left(\widehat{q}_j^{(1)}, \widehat{M}_{jk}^{(1)}, \widehat{\mathcal{J}}^{(1)}, \widehat{p}_j^{(1)0}\right) = \mathcal{F}\left(q_j^{(1)}, M_{jk}^{(1)}, \mathcal{J}^{(1)}, p_j^{(1)}\right)\mathcal{F}^{-1},$$
$$\left(\widehat{q}_j^{(2)}, \widehat{M}_{jk}^{(2)}, \widehat{\mathcal{J}}^{(2)}, \widehat{p}_j^{(2)}\right) = \mathcal{F}\left(q_i^{(2)}, M_{ij}^{(2)}, \mathcal{J}^{(2)}, p_i^{(2)}\right)\mathcal{F}^{-1}, \tag{21}$$

we have

$$\widehat{q}_j^{(1)} = \mathrm{i}\ell\left(k_w\frac{\partial}{\partial k_j} - k_j\frac{\partial}{\partial k_w}\right), \qquad \widehat{M}_{jk}^{(1)} = -\mathrm{i}\left(k_j\frac{\partial}{\partial k_k} - k_k\frac{\partial}{\partial k_j}\right),$$
$$\widehat{\mathcal{J}}^{(1)} = \ell k_w, \qquad\qquad\qquad \widehat{p}_j^{(1)} = \hbar k_j, \tag{22}$$

while

$$
\widehat{q}_j^{(2)} = \mathrm{i}\frac{\partial}{\partial k_j} + \mathrm{i}\ell\left(k_w\frac{\partial}{\partial k_j} - k_j\frac{\partial}{\partial k_w}\right), \qquad \widehat{M}_{jk}^{(2)} = -\mathrm{i}\left(k_j\frac{\partial}{\partial k_k} - k_k\frac{\partial}{\partial k_j}\right),
$$
$$
\widehat{\mathcal{J}}^{(2)} = I + \ell k_w, \qquad\qquad\qquad \widehat{p}_j^{(2)} = \hbar k_j, \tag{23}
$$

so that in equations (23), as in equations (19), a representation of the Heisenberg algebra survives in the $\ell \to 0$ limit. Of course the spectra of the self-adjoint operators representing the generators of the deformed algebra are the same across all unitarily equivalent representations.

We focus now for simplicity on the case $d = 1$, which illustrates well the issues discussed in this paper. Because the effective spatial dimension for the local current algebra will be $d + 1$, there is also the interesting possibility for $d = 1$ that features of anyonic statistics of point particles in two-space could occur. We return to this point in the discussion below.

For $d = 1$, the algebra of equations (6) reduces to

$$
[q, p] = \mathrm{i}\hbar\mathcal{J}, \qquad [q, \mathcal{J}] = -\mathrm{i}\frac{\ell^2}{\hbar}p, \qquad [p, \mathcal{J}] = 0. \tag{24}
$$

In the representation of equations (22), it is useful to introduce polar coordinates (ρ, ψ), with $k_x = k_1 = \rho\sin\psi$, $k_w = \rho\cos\psi$ and $\mathrm{d}k_x\,\mathrm{d}k_w = \rho\,\mathrm{d}\rho\,\mathrm{d}\psi$. Then the operators become

$$
\widehat{q}^{(1)} = \mathrm{i}\ell\frac{\partial}{\partial\psi}, \qquad \widehat{p}^{(1)} = \hbar\rho\sin\psi, \qquad \widehat{\mathcal{J}}^{(1)} = \ell\rho\cos\psi. \tag{25}
$$

The representation by equations (25) in $\widehat{\mathcal{H}}$ is obviously reducible, since the subspace of momentum–space wavefunctions $\widehat{\Phi}$ with support between ρ_0 and $\rho_0 + \Delta\rho$ is invariant under these operators. Indeed, the Casimir operator

$$
C = \frac{1}{\hbar^2}p^2 + \frac{1}{\ell^2}\mathcal{J}^2 \tag{26}
$$

commutes with all of the generators in equations (24). The corresponding operator $\widehat{C}$ defined from equations (25) acts in $\widehat{\mathcal{H}}$ via multiplication by ρ^2. The eigenvalues ρ_0^2 of C label unitarily inequivalent irreducible representations of (24), and we see that the reducible representation given by equations (25) acting in $\widehat{\mathcal{H}}$ is actually a direct integral (from $\rho_0 = 0$ to $+\infty$) of irreducible representations. The irreducible component associated with the eigenvalue ρ_0^2 consists of operators acting on the Hilbert space $\widehat{\mathcal{H}}_{\rho_0}$ of complex-valued functions $\widehat{\Phi}_{\rho_0}(\psi)$ on the circle of radius ρ_0 centred at the origin in (k_x, k_w)-space that are square-integrable with respect to the measure $\mathrm{d}\psi$.

Let us denote by $\widehat{q}_{\rho_0}^{(1)}$ the self-adjoint operator $\mathrm{i}\ell\partial/\partial\psi$ acting in $\widehat{\mathcal{H}}_{\rho_0}$ (defined on a domain of essential self-adjointness that includes everywhere essentially continuous and differentiable functions). Let $\widehat{p}_{\rho_0}^{(1)}$ and $\widehat{\mathcal{J}}_{\rho_0}^{(1)}$ be, respectively, the multiplication operators $\hbar\rho_0\sin\psi$ and $\ell\rho_0\cos\psi$ acting in $\widehat{\mathcal{H}}_{\rho_0}$. Then a complete orthogonal basis for $\widehat{\mathcal{H}}_{\rho_0}$ is provided by the eigenfunctions of $\widehat{q}_{\rho_0}^{(1)}$, specifically $\left\{\widehat{\Phi}_{\rho_0}^{(n)}(\psi) = \mathrm{e}^{-\mathrm{i}n\psi} : n \in \mathbb{Z}\right\}$, which may be regarded as infinitely differentiable functions on the circle of radius $\rho = \rho_0$. Thus $\widehat{q}_{\rho_0}^{(1)}$ has the eigenvalue spectrum $\{n\ell : n \in \mathbb{Z}\}$, and the positional spectrum (which is invariant under unitary equivalence) is discrete and unbounded in an irreducible self-adjoint representation of the deformed algebra. On the other hand, the spectra of $\widehat{p}_{\rho_0}^{(1)}$ and $\widehat{\mathcal{J}}_{\rho_0}^{(1)}$ are continuous and bounded in absolute value by $\hbar\rho_0$, and $\ell\rho_0$, respectively, in such an irreducible representation.

Note that the complex-valued functions on the plane given by $\widehat{\Phi}^{(n)}(\rho, \psi) = \mathrm{e}^{-\mathrm{i}n\psi}$, $n \in \mathbb{Z}$, do not belong to $\widehat{\mathcal{H}}$ as they are not square-integrable with respect to $\rho\,\mathrm{d}\rho\,\mathrm{d}\psi$. These are 'non-normalizable' eigenfunctions of the operator $\widehat{q}^{(1)}$ of equation (25). *Bona fide* square-integrable eigenfunctions of $\widehat{q}^{(1)}$ in $\widehat{\mathcal{H}}$ take the form $f(\rho)\,\mathrm{e}^{-\mathrm{i}n\psi}$, where $\int |f(\rho)|^2\rho\,\mathrm{d}\rho$ is finite.

One regards the operator $\widehat{q}^{(1)}$ as the direct integral over ρ of the measurable field of operators $\widehat{q}_\rho^{(1)}$ and the corresponding direct integral structure of the Hilbert space is given by

$$\widehat{\mathcal{H}} = \int_0^\infty \widehat{\mathcal{H}}_\rho \rho \, \mathrm{d}\rho. \tag{27}$$

Now, in the representations of equations (24) given by equations (17) or (19) acting in $\mathcal{H}$, we have corresponding decompositions into direct integrals of irreducible representations over the parameter ρ [33]. The irreducible representations act in Hilbert spaces $\mathcal{H}_\rho$, and the operator $q^{(1)}$ (respectively, $q^{(2)}$) is the direct integral over ρ of operators $q_\rho^{(1)}$ (respectively, $q_\rho^{(2)}$) that act in $\mathcal{H}_\rho$. Let us introduce polar coordinates (r, θ), with $x = r \sin\theta$ and $w = r \cos\theta$. Then the corresponding eigenfunctions of the operator $q_{\rho_0}^{(1)}$, with eigenvalues $n\ell$, in an irreducible representation labelled by ρ_0, are the functions $\{\Phi_{\rho_0}^{(n)}(r, \theta) = \mathrm{e}^{\mathrm{i}n\theta} : n \in \mathbb{Z}\}$. The corresponding eigenfunctions of the operator $q_{\rho_0}^{(2)}$ are $\{\mathrm{e}^{\mathrm{i}n\theta} \mathrm{e}^{-\mathrm{i}(r/\ell)\cos\theta} : n \in \mathbb{Z}\}$. The associated decomposition of $\mathcal{H}$ as a direct integral of $\mathcal{H}_\rho$ with respect to $\rho \, \mathrm{d}\rho$, is developed in [33] using the Fourier–Bessel transformation. For any element $\Phi(x, w)$ of $\mathcal{H}$, we have $\Phi(r \sin\theta, r \cos\theta) = \int_0^\infty \Phi_\rho(r \sin\theta, r \cos\theta)\rho \, \mathrm{d}\rho$, where

$$\Phi_\rho(r \sin\theta, r \cos\theta) = (2\pi)^{-1} \int_0^{2\pi} \mathrm{d}\theta' \int_0^\infty r' \, \mathrm{d}r' \Phi(r' \sin\theta', r' \cos\theta')$$
$$\times J_0\left[\rho\sqrt{r^2 + (r')^2 - 2rr'\cos(\theta - \theta')}\right]. \tag{28}$$

Finally, we can explore the rather subtle way that the usual, irreducible representation of the Heisenberg algebra must be recovered from irreducible representations of the deformed algebra (24), in the limit $\ell \to 0$, a point that was not addressed by Vilela Mendes. With $[\mathcal{F}\Phi](k_x, k_w) = \widehat{\Phi}(k_x, k_w)$, it is easy to show that $[\mathcal{F}(U_\ell\Phi)](k_x, k_w) = \widehat{\Phi}(k_x, k_w + 1/\ell)$. Thus, considering the representation in $\widehat{\mathcal{H}}$ obtained from equations (23) (with $d = 1$), the irreducible components contributing to the direct integral are defined from wavefunctions having support on circles centred at the point $k_x = 0$, $k_w = -1/\ell$ in (k_x, k_w)-space, i.e.

$$k_x^2 + \left(k_w + \frac{1}{\ell}\right)^2 = \rho_0^2. \tag{29}$$

To obtain the usual representation of the Heisenberg algebra as $\ell \to 0$, we must arrive at wavefunctions in the limiting representation that become independent of k_w. However, from equation (29), we see that if we try to let $\ell \to 0$ while ρ_0 is held fixed, we obtain no such limit, rather, $|k_w|$ becomes arbitrarily large while the operator $p^{(2)}$ (which acts through multiplication by $\hbar k_x$) remains bounded. The way out of this difficulty is to allow ρ_0 to depend on ℓ. Taking $\rho_0 = 1/\ell$ in (29), we have for a very small ℓ a circle of very large radius tangent to the horizontal axis at the origin, approximating the line $k_w = 0$, indeed,

$$k_w = -\frac{1}{\ell} + \frac{1}{\ell}\left(1 - k_x^2\ell^2\right)^{\frac{1}{2}}, \tag{30}$$

so that for any fixed value of k_x, $k_w \approx -(1/2)k_x^2\ell \to 0$ in the limit as $\ell \to 0$. Thus, traversing a parameterized family of irreducible representations labelled by $\rho_0 = 1/\ell$ (or at least, having the property that ρ_0 tends toward $1/\ell$ as $\ell \to 0$) is the appropriate way to obtain the Heisenberg algebra in the limiting irreducible representation.

Moreover, the condition $\rho_0 = 1/\ell$ allows the operator $p^{(2)}$ (whose spectrum is bounded by $\hbar\rho_0$) to become unbounded as desired when $\ell \to 0$.

4. The kinetic energy and harmonic oscillator Hamiltonians

To investigate in detail the quantum-mechanical behaviour of a particle system described by the deformed algebra, such as the harmonic oscillator with a fundamental length scale, we need to settle on the form of the kinetic energy part of the Hamiltonian H_0, and write the total Hamiltonian $H = H_0 + V$. In [8], it was suggested that for a particle of mass $\mathfrak{m}$, we should use $H_0 = p^2/2\mathfrak{m}$, where p is the generator appearing in the algebra of equations (24), and that the oscillator Hamiltonian should then be $H_{\mathrm{osc}} = p^2/2\mathfrak{m} + \mathfrak{m}\omega^2 q^2/2$.

But one reasonable criterion for determining the choice of H_0 is the physical condition that the time-derivative of the particle position should be the particle velocity. That is, we should expect H_0 and H_{osc} to satisfy

$$\dot{q} = \frac{1}{i\hbar}[q, H_0] = \frac{1}{i\hbar}[q, H_{\mathrm{osc}}] = \frac{p}{\mathfrak{m}}. \tag{31}$$

However, we have from equations (24) that

$$\frac{1}{i\hbar}\left[q, \frac{p^2}{2\mathfrak{m}}\right] = \frac{1}{2\mathfrak{m}}(p\mathcal{J} + \mathcal{J}p), \tag{32}$$

which becomes $p/\mathfrak{m}$ when $\mathcal{J}$ is the identity operator, but not otherwise. To fulfill equation (31), we propose to modify the form of the kinetic energy term in the Hamiltonian, so that

$$H_0 = \frac{1}{2\mathfrak{m}}\left\{p^2 + \frac{\hbar^2}{\ell^2}(\mathcal{J} - I)^2\right\}, \tag{33}$$

where I is the identity operator in a representation of the algebra. Note that the coefficient $\hbar^2/\ell^2$ in (33) is needed to obtain the correct commutation relation with q.

Now, in the representation $q^{(2)}, p^{(2)}, \mathcal{J}^{(2)}$ of the deformed algebra given by equations (19), with $d = 1$, we have

$$H_0^{(2)} = -\frac{\hbar^2}{2\mathfrak{m}}\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial w^2}\right\}, \tag{34}$$

while in the Fourier-transformed representation $\widehat{q}^{(2)}, \widehat{p}^{(2)}, \widehat{\mathcal{J}}^{(2)}$, we have

$$\widehat{H}_0^{(2)} = \frac{\hbar^2}{2\mathfrak{m}}\left(k_x^2 + k_w^2\right). \tag{35}$$

The explicit dependence of H_0 on ℓ has disappeared.

Nevertheless, this is the family of irreducible representations that goes over smoothly to the standard representations of the Heisenberg algebra as we take the limit $\ell \to 0$, with $\rho_0 = 1/\ell$. Returning to the discussion following equation (30), we have from equation (35),

$$\widehat{H}_0^{(2)} \approx (\hbar^2/2\mathfrak{m})\left[k_x^2 + (\ell^2/4)k_x^4\right], \tag{36}$$

so that $\widehat{H}_0^{(2)} \to (\hbar^2/2\mathfrak{m})k_x^2$ as $\ell \to 0$.

Now the oscillator potential, and other potentials of the form $V(q)$, commute with q, so that $[q, H_0 + V] = [q, H_0]$, and it is appropriate to take the harmonic oscillator Hamiltonian for the deformed quantum mechanics to be $H_{\mathrm{osc}} = H_0 + \mathfrak{m}\omega^2 q^2/2$, with H_0 as in equation (33). From (19), however, we see that the potential energy term no longer acts via multiplication in $L_{\mathrm{d}x\,\mathrm{d}w}^2$, but as a differential operator via further derivative terms.

5. Discretized and extended local current algebras

In this section we explore two approaches to the introduction of local currents.

In the first approach, we interpret locality with respect to a basis of eigenvectors of q in an irreducible representation of the deformed Heisenberg algebra (24). This leads to discretized mass and momentum density operators, i.e. a theory on a lattice.

In the second approach, we interpret 'locality' with respect to the (xw)-space on which representations of (24) are modelled. Here we consider two possibilities. The first is to work straightforwardly with the nonrelativistic current algebra in two-dimensional space (LCA2) to describe the one-dimensional, deformed quantum kinematics, while the second is to introduce a semidirect sum of LCA2 with the algebra of vector fields on the line. In the discussion, we note the tension occurring between locality and irreducibility.

5.1. Locality with respect to the discrete positional spectrum

In an irreducible representation of equations (25), the spectrum of the self-adjoint operator representing the generator q is discrete, given by $n\ell$ for $n \in \mathbb{Z}$; write the corresponding eigenvector as $|n\ell\rangle$. We shall then write $q|n\ell\rangle = n\ell|n\ell\rangle$, and

$$q = \sum_{n=-\infty}^{\infty} n\ell|n\ell\rangle\langle n\ell|, \qquad I = \sum_{n=-\infty}^{\infty} |n\ell\rangle\langle n\ell|. \tag{37}$$

The corresponding local mass density operator $\mathfrak{J}_q$ takes the form

$$\mathfrak{J}_q(g) = \mathfrak{m} \sum_{n=-\infty}^{\infty} g(n\ell)|n\ell\rangle\langle n\ell|, \tag{38}$$

where by analogy with the continuum case, the real-valued function g has compact support, i.e. for some $N > 0$, $g(n\ell) = 0$ whenever $|n\ell| > N\ell$. When $g(n\ell)$ approximates the function $n\ell$, $\mathfrak{J}_q(g)$ approximates the moment operator $\mathfrak{m}q$. When $g(n\ell) \geqslant 0$ ($\forall n \in \mathbb{Z}$), $\mathfrak{J}_q(g)$ is a positive operator. When $g(n\ell)$ approximates the constant function 1, $\mathfrak{J}_q(g)$ approximates the mass times the identity operator. As we are in a representation of the local currents describing a single particle, we can interpret $(1/\mathfrak{m})\mathfrak{J}_q(g)$ as a spatial probability density operator averaged with $g(n\ell)$.

Equation (25) implies

$$\langle n\ell|p|m\ell\rangle = \frac{\hbar\rho_0}{2\mathrm{i}}(\delta_{n+1,m} - \delta_{n-1,m}) \tag{39}$$

and

$$\langle n\ell|\mathcal{J}|m\ell\rangle = \frac{\ell\rho_0}{2}(\delta_{n+1,m} + \delta_{n-1,m}). \tag{40}$$

Introduce the local currents

$$\mathfrak{J}_p(h) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \widetilde{h}(n\ell)\{p|n\ell\rangle\langle n\ell| + |n\ell\rangle\langle n\ell|p\} \tag{41}$$

and

$$\mathfrak{J}_\mathcal{J}(r) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \widetilde{r}(n\ell)\{\mathcal{J}|n\ell\rangle\langle n\ell| + |n\ell\rangle\langle n\ell|\mathcal{J}\}, \tag{42}$$

where $h(n\ell) \equiv (1/2)[\widetilde{h}(n\ell)+\widetilde{h}((n+1)\ell)]$ and $r(n\ell) \equiv (1/2)[\widetilde{r}(n\ell)+\widetilde{r}((n+1)\ell)]$ are also taken to be compactly supported. As $\widetilde{h}(n\ell)$ and $\widetilde{r}(n\ell)$ approximate the function that is identically 1,

so do $h(n\ell)$ and $r(n\ell)$, then $\mathfrak{J}_p(h)$ approximates p, $\mathfrak{J}_{\mathcal{J}}(r)$ approximates $\mathcal{J}$, and the global algebra is recovered.

From equations (39)–(42), we find

$$\mathfrak{J}_p(h) = \frac{\hbar\rho_0}{2\mathrm{i}} \sum_{n=-\infty}^{\infty} h(n\ell)\{|n\ell\rangle\langle(n+1)\ell| - |(n+1)\ell\rangle\langle n\ell|\}, \tag{43}$$

$$\mathfrak{J}_{\mathcal{J}}(r) = \frac{\ell\rho_0}{2} \sum_{n=-\infty}^{\infty} r(n\ell)\{|n\ell\rangle\langle(n+1)\ell| + |(n+1)\ell\rangle\langle n\ell|\}. \tag{44}$$

For the Lie algebra of currents generated by these operators, in the irreducible representation labelled by ρ_0, to be *local,* we need the commutator brackets of the operators $\mathfrak{J}_q(g)$, $\mathfrak{J}_p(h)$ and $\mathfrak{J}_{\mathcal{J}}(r)$ given by equations (38), (43) and (44) to yield similarly local expressions. These expressions are all linear combinations of operators of the form $|n\ell\rangle\langle n\ell|$, $|n\ell\rangle\langle(n+1)\ell|$ and $|(n+1)\ell\rangle\langle n\ell|$.

In fact, we have

$$[\mathfrak{J}_q(g_1), \mathfrak{J}_q(g_2)] = 0, \tag{45}$$

$$[\mathfrak{J}_q(g), \mathfrak{J}_p(h)] = -\mathrm{i}\frac{\mathfrak{m}\hbar}{\ell}\mathfrak{J}_{\mathcal{J}}(r), \tag{46}$$

where $r(n\ell) = h(n\ell)\{g(n\ell) - g([n+1]\ell)\}$, and

$$[\mathfrak{J}_q(g), \mathfrak{J}_{\mathcal{J}}(r)] = \mathrm{i}\frac{\mathfrak{m}\ell}{\hbar}\mathfrak{J}_p(h), \tag{47}$$

where $h(n\ell) = r(n\ell)(g(n\ell) - g([n+1]\ell))$, which thus far are satisfactorily local. But other commutators, such as $[\mathfrak{J}_p(h), \mathfrak{J}_{\mathcal{J}}(r)]$, generate terms of the form $|(n+1)\ell\rangle\langle(n-1)\ell|$ and $|(n-1)\ell\rangle\langle(n+1)\ell|$, so that successive commutators generate additional terms $|(n-m)\ell\rangle\langle(n+m)\ell|$ and $|(n+m)\ell\rangle\langle(n-m)\ell|$, for arbitrary $m \in \mathbb{Z}$. Therefore, to close the Lie algebra of these currents, one is forced to include new basis elements in the (already infinite-dimensional) current algebra, having more general forms, e.g.

$$\sum_{n,m=-\infty}^{\infty} s(n\ell, m\ell)\{|(n+m)\ell\rangle\langle(n-m)\ell| \pm |(n-m)\ell\rangle\langle(n+m)\ell|\},$$

where s is a compactly supported function on the square lattice of points $(n\ell, m\ell)$. Such currents are *nonlocal* in the positional eigenvalues, since $(n-m)\ell$ and $(n+m)\ell$ become arbitrarily far apart. This sort of behaviour by the commutation relations of discretized local derivatives is well known in the context of lattice models.

Before leaving the discussion of the discretized current algebra, it is worth remarking that we do have within this framework an *equation of continuity* for the deformed quantum theory, relating the time derivative of $\mathfrak{J}_q$ to the spatial divergence of $\mathfrak{J}_p$. Taking the Hamiltonian H to be $H_0 + V(q)$, with H_0 given by equation (33), we have

$$\dot{\mathfrak{J}}_q(g) = \frac{1}{\mathrm{i}\hbar}[\mathfrak{J}_q(g), H] = \frac{\mathfrak{m}}{\mathrm{i}\hbar}\left[\sum_{n=-\infty}^{\infty} g(n\ell)|n\ell\rangle\langle n\ell|, H\right]$$

$$= \frac{\mathfrak{m}}{\mathrm{i}\hbar}\left[\sum_{n=-\infty}^{\infty} g(n\ell)|n\ell\rangle\langle n\ell|, H_0\right]. \tag{48}$$

Straightforward calculations yield

$$\left[\sum_n g(n\ell)|n\ell\rangle\langle n\ell|, p^2\right] = \left(\frac{\hbar\rho_0}{2\mathrm{i}}\right)^2 \sum_n g(n\ell)\{-|(n+2)\ell\rangle\langle n\ell| + |n\ell\rangle\langle(n+2)\ell|\}$$

$$+ \left(\frac{\hbar\rho_0}{2\mathrm{i}}\right)^2 \sum_n g(n\ell)\{-|(n-2)\ell\rangle\langle n\ell| + |n\ell\rangle\langle(n-2)\ell|\}$$

$$= \left(\frac{\hbar\rho_0}{2\mathrm{i}}\right)^2 \sum_n (g((n-2)\ell) - g(n\ell))$$

$$\cdot \{|(n-2)\ell\rangle\langle n\ell| - |n\ell\rangle\langle(n-2)\ell|\}, \tag{49}$$

while

$$\left[\sum_n g(n\ell)|n\ell\rangle\langle n\ell|, (\mathcal{J} - I)^2\right] = -\frac{\ell^2\rho_0^2}{4} \sum_n g(n\ell)\{|(n-2)\ell\rangle\langle n\ell| - |n\ell\rangle\langle(n-2)\ell|\}$$

$$- \frac{\ell^2\rho_0^2}{4} \sum_n g(n\ell)\{|(n+2)\ell\rangle\langle n\ell| - |n\ell\rangle\langle(n+2)\ell|\}$$

$$+ \ell\rho_0 \sum_n g(n\ell) \cdot \{|(n-1)\ell\rangle\langle n\ell| - |n\ell\rangle\langle(n-1)\ell|\}$$

$$+ \ell\rho_0 \sum_n g(n\ell) \cdot \{|(n+1)\ell\rangle\langle n\ell| - |n\ell\rangle\langle(n+1)\ell|\}$$

$$= -\frac{\ell^2\rho_0^2}{4} \sum_n (g(n\ell) - g((n-2)\ell))$$

$$\cdot \{-|n\ell\rangle\langle(n-2)\ell| + |(n-2)\ell\rangle\langle n\ell|\}$$

$$+ \ell\rho_0 \sum_n (g(n\ell) - g((n-1)\ell))$$

$$\cdot \{-|n\ell\rangle\langle(n-1)\ell| + |(n-1)\ell\rangle\langle n\ell|\}.$$

Then from equation (48), with H_0 as in equation (33), we obtain (after replacing the index n by $n+1$ in the infinite sum)

$$\dot{\mathfrak{J}}_q(g) = \frac{\hbar\rho_0}{2\mathrm{i}} \sum_n \frac{(g((n+1)\ell) - g(n\ell))}{\ell} \cdot \{-|(n+1)\ell\rangle\langle n\ell| + |n\ell\rangle\langle(n+1)\ell|\} = \mathfrak{J}_p(Dg), \tag{50}$$

where

$$Dg(n\ell) \equiv \frac{g((n+1)\ell) - g(n\ell)}{\ell} \tag{51}$$

is the discretized derivative. Evidently, equation (50) is precisely the required continuity equation. The density $\mathfrak{J}_q$ and current $\mathfrak{J}_p$ that appear in this equation of continuity are local, but they belong to a Lie algebra that necessarily includes currents that are nonlocal with respect to the positional operator q.

5.2. Locality with respect to the extended spatial manifold

Consider again the global Lie algebra given by equations (24). A quite different approach to introducing local currents is suggested by the form of the representation of this Lie algebra by equations (19) with $d = 1$. The idea is to define a current algebra that is local in (x, w)-space,

from which—with the right choices of limiting test functions—we shall be able to recover equations (19).

Thus, let us refer back to the LCA of equations (12), and interpret these equations as applying in a two-dimensional Euclidean space with coordinates (x, w), extending the spatial manifold by one dimension. We then have the operator-valued distribution $Q(h, g_x, g_w)$ acting in $L^2_{dx\,dw}$, where h is drawn from the space of compactly-supported, real-valued C^∞ test functions on (x, w)-space, and g_x, g_w are the components of a compactly-supported, C^∞ vector field on (x, w)-space

$$Q(h, g_x, g_w) = h(x, w) + \frac{1}{2i} \left\{ g_x(x, w) \frac{\partial}{\partial x} + \frac{\partial}{\partial x} g_x(x, w) \right\}$$
$$+ \frac{1}{2i} \left\{ g_w(x, w) \frac{\partial}{\partial w} + \frac{\partial}{\partial w} g_w(x, w) \right\}. \tag{52}$$

Defining $Q_\ell(h, g_x, g_w) = Q(h, \ell g_x, \ell g_w)$ for $\ell > 0$, we obtain a family of operators parameterized by ℓ. In the $\ell \to 0$ limit, $Q_\ell(h, g_x, g_w)$ reduces to $Q(h, 0, 0)$. Then with

$$\rho(f) = \lim_{h \to f} \mathfrak{m} Q(h, 0, 0), \tag{53}$$

we recover the earlier one-particle mass density operator in one space dimension. The limit here pertains to the fact that f depends only on x and is independent of w, while h is compactly supported in (x, w)-space.

The local current $Q_\ell(h, g_x, g_w)$ is clearly motivated by the form of $q^{(2)}$ in equation (19); in fact, in the limit where $h(x, w)$ approaches the coordinate function x, and the vector field $(g_x(x, w), g_w(x, w))$ approaches $(-w, x)$, we recover $q^{(2)}$ with the space dimension $d = 1$. Evidently, Q (or, alternatively, Q_ℓ) is also sufficiently general to let us obtain the other global operators in the deformed current algebra, when suitable limits of h, g_x and g_w are taken. Thus the operator $p^{(2)}$ (with $d = 1$) is just $Q(0, \hbar g_x, 0)$ or $Q_\ell(0, (\hbar/\ell) g_x, 0)$, taken in the limit where g_x approaches the constant vector field of magnitude 1. Likewise $\mathcal{J}^{(2)}$ is $Q(h, 0, \ell g_w)$ or $Q_\ell(h, 0, g_w)$, taken in the limit where both h and g_w become identically 1.

The natural choices of local currents corresponding to $p^{(2)}$ and $\mathcal{J}^{(2)}$ are, respectively,

$$J(g) = \frac{\hbar}{2i} \left\{ g(x) \frac{\partial}{\partial x} + \frac{\partial}{\partial x} g(x) \right\} \tag{54}$$

and

$$\mathcal{J}(k) = k(w) + \frac{\ell}{2i} \left\{ k(w) \frac{\partial}{\partial w} + \frac{\partial}{\partial w} k(w) \right\}, \tag{55}$$

where $g(x)$ and $k(w)$ are compactly-supported C^∞ functions on $\mathbb{R}$. These local currents incorporate the intuitive idea of local flows in the two coordinate directions. To express them in terms of Q or Q_ℓ, we write (again recalling that the arguments of Q are compactly supported in both the x and w coordinates)

$$J(g) = \lim_{g_x \to g} Q(0, \hbar g_x, 0) = \lim_{g_x \to g} Q_\ell\left(0, \frac{\hbar}{\ell} g_x, 0\right) \tag{56}$$

and

$$\mathcal{J}(k) = \lim_{h \to k} \lim_{g_w \to k} Q(h, 0, \ell g_w) = \lim_{h \to k} \lim_{g_w \to k} Q_\ell(h, 0, g_w). \tag{57}$$

Because of the way the operator $q^{(2)}$ mixes the x and w directions, it is necessary to incorporate the full (x, w)-dependence in the test functions h, g_x and g_w that appear as arguments of Q. Then the current algebra that accommodates all the natural local and global limits is just LCA2, i.e. the algebra of $Q(h, g_x, g_w)$ satisfying the semidirect sum Lie algebra of equation (13) with

$d = 2$. So we have the *usual* local current algebra of nonrelativistic quantum mechanics, but localized in two space dimensions rather than just one.

An interesting feature of this framework is that the decomposition of $L^2_{\mathrm{d}x\mathrm{d}w}(\mathbb{R}^2)$ into a direct integral of irreducible representations of the global algebra (labelled, as in section 3, by ρ_0), is not respected by the one-particle irreducible representation of the LCA. That is, the *local* currents unavoidably *connect* the reducing subspaces of the global algebra. We have a kind of tension between our desire to incorporate local currents, and the assumption that we can work with a single, irreducible representation of the global, deformed algebra.

This situation does not occur for the usual Heisenberg algebra, where the Hilbert space for a single irreducible representation labelled by $\hbar$ also carries the one-particle representation of the full LCA. However, it is reminiscent of earlier results pertaining to self-adjoint representations of the LCA describing *spinning* particles in three space dimensions. Here irreducible representations of the global algebra describe quantum particles with fixed spin— i.e., the operators act within a single irreducible representation of $SU(2)$—while the local currents inevitably contain spin-changing terms, that connect representations associated with a tower of different spins [27].

Let us take another look at how we can recover LCA1 as the $\ell \to 0$ limit of LCA2 in the single-particle representation written above. If we take the operators Q_ℓ as our starting point, with fixed test functions (h, g_x, g_w) independent of ℓ, then as $\ell \to 0$ and $h \to f$, we recover only the density operator $\rho(f)$, not the full LCA1. To recover the local currents $J(g)$ in the $\ell \to 0$ limit, we must allow at least some of the test functions themselves to be ℓ-dependent, as in equation (56). But the form of $J(g)$ in equation (54) is actually independent of ℓ. This suggests that, for a given ℓ, we consider an *extension* of the current algebra LCA2 (generated by the $Q_\ell(h, g_x, g_w)$) by the algebra of vector fields on the line (generated by the $J(g)$ in equation (54)), via the bracket

$$[Q_\ell(h, g_x, g_w), J(g)] = \mathrm{i}\hbar Q_\ell(\widetilde{h}, \widetilde{g}_x, \widetilde{g}_w), \tag{58}$$

where

$$\begin{aligned}
\widetilde{h}(x, w) &= g(x)\frac{\partial}{\partial x}h(x, w), \\
\widetilde{g}_x(x, w) &= g(x)\frac{\partial}{\partial x}g_x(x, w) - g_x(x, w)\frac{\partial}{\partial x}g(x), \\
\widetilde{g}_w(x, w) &= g(x)\frac{\partial}{\partial x}g_w(x, w).
\end{aligned} \tag{59}$$

Note in equations (58) and (59) that $h, \widetilde{h}$ are compactly-supported, C^∞ functions on (x, w)-space, (g_x, g_w) and $(\widetilde{g}_x, \widetilde{g}_w)$ are compactly-supported, C^∞ vector fields on (x, w)-space, while g is a compactly-supported, C^∞ vector field in the x-coordinate only.

Now in the limit $\ell \to 0$, $Q_\ell(h, g_x, g_w)$ becomes multiplication by $h(x, w)$, while $J(g)$ survives as the operator for total momentum density in the x-direction, independent of (or integrated over) w. The representation is still reducible; w has become a kind of unobservable, internal coordinate for a particle theory in one space dimension. This construction also generalizes to higher space dimensions, augmented by the one additional coordinate w. Note, however, that the form of the kinetic energy term in the Hamiltonian, given by $H_0^{(2)}$ in equation (34), is *independent* of ℓ; the second derivative with respect to w does not vanish as $\ell \to 0$. Upon taking this limit, we can use the fact that the operators $\rho(f)$ and $J(g)$ commute with $\partial^2/\partial w^2$ to recover the continuity equation in the continuum

$$\dot{\rho}(f) = \frac{1}{\mathrm{i}\hbar}[\rho(f), H_0] = J\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right), \tag{60}$$

and the ordinary quantum mechanics of a free particle having x as its positional coordinate.

For local currents as defined by equation (52) in (x, w)-space, we can also write an equation of continuity for a free particle,

$$\mathfrak{m}\dot{Q}(h, 0, 0) = \frac{\mathfrak{m}}{\mathrm{i}\hbar}[Q(h, 0, 0), H_0] = \hbar Q\left(0, \frac{\partial h}{\partial x}, \frac{\partial h}{\partial w}\right). \tag{61}$$

But note that $Q(h, 0, 0)$ is no longer a *positional* mass density. If the potential energy V is a function of the (deformed) position operator q (as in the case of the deformed harmonic oscillator), then $V(q)$ does not commute with $\mathfrak{m}\dot{Q}(h, 0, 0)$ and we have no such continuity equation.

6. Concluding remarks

In the case of one space dimension, we have described two approaches to introducing fixed-time local currents for a subalgebra of the deformed Poincaré–Heisenberg algebra discussed by Vilela Mendes. The first requires a nonlocal Lie algebra generated by discretized local currents, but the currents act within an irreducible representation of the global algebra. The second requires adjoining an extra dimension to the spatial manifold, and the local currents connect the reducing subspaces in a direct integral of irreducible representations of the global algebra.

From our perspective, the latter approach offers some attractive possibilities. We have mentioned above the existence of many interesting, inequivalent representations of LCA2, including representations describing N particles satisfying the statistics of anyons. This suggests a possible new interpretation of such representations—not as describing conventional particles in two-space, but as describing local currents for a deformed algebra of quantum mechanics having some anyonic properties. Such an interpretation is a topic of continuing investigation by the authors.

Acknowledgments

References

[1] Snyder H S 1947 *Phys. Rev.* **71** 38
   Snyder H S 1947 *Phys. Rev.* **72** 68
[2] Yang C N 1947 *Phys. Rev.* **72** 874
[3] Connes A 1994 *Noncommutative Geometry* (New York: Academic)
[4] Douglas M R and Hull C M 1998 *J. High Energy Phys.* JHEP02008(1998)
[5] Ardalan F, Arfaei H and Sheik-Jabbari M M 1999 *J. High Energy Phys.* JHEP02(1999)
[6] Mavromatos N E and Szabo R J 1999 *Phys. Rev.* D **59** 104018
[7] Arkani-Hamed N, Dimopoulos S and Dvali G 1998 *Phys. Lett.* B **429** 263
[8] Vilela Mendes R 2000 *J. Math. Phys.* **41** 156
[9] Gerstenhaber M 1964 *Ann. Math.* **79** 59
[10] Levy-Nahas M 1967 *J. Math. Phys.* **8** 1211
[11] Nijenhuis A and Richardson R W 1967 *J. Math. Mech.* **17** 89
[12] Vilela Mendes R 1994 *J. Phys. A: Math. Gen.* **27** 8091–104
[13] de Azcarraga J A and Izquierdo J M 1995 *Lie Groups, Lie Algebras, Cohomology and Some Applications in Physics* (Cambridge: Cambridge University Press)
[14] Chryssomalakos C and Okon E 2004 *Int. J. Mod. Phys.* D **13** 2003 (*Preprint* hep-th/0410212)

[15] Barut A O and Bohm A *J. Math. Phys.* **11** 2938–45
[16] Carlen E and Vilela Mendes R 2001 *Phys. Lett.* A **290** 109
[17] Douglas M R and Nekrasov N A 2001 *Rev. Mod. Phys.* **73** 977
[18] Szabo R 2001 Quantum field theory on noncommutative spaces *Preprint* hep-th/0109162
[19] Dashen R and Sharp D H 1968 *Phys. Rev.* **165** 1857
[20] Goldin G A 1971 *J. Math. Phys.* **12** 462
[21] Leinaas J M and Myrheim J 1977 *Nuovo Cimento* B **37** 1
[22] Goldin G A, Menikoff R and Sharp D H 1980 *J. Math. Phys.* **21** 650
[23] Goldin G A, Menikoff R and Sharp D H 1981 *J. Math. Phys.* **22** 1664
[24] Wilczek F 1982 *Phys. Rev. Lett.* **48** 1144
[25] Goldin G A and Sharp D H 1983 *Phys. Rev.* D **28** 830
[26] Goldin G A, Menikoff R and Sharp D H 1983 *Phys. Rev. Lett.* **51** 2246
[27] Goldin G A and Sharp D H 1983 *Commun. Math. Phys.* **92** 217
[28] Goldin G A and Menikoff R 1985 *J. Math. Phys.* **25** 1880
[29] Goldin G A, Grodnik J, Powers R T and Sharp D H 1974 *J. Math. Phys.* **15** 88
[30] Albeverio S, Kondratiev Yu G and Rockner L M 1999 *Rev. Math. Phys.* **11** 1
[31] Goldin G A 2004 Lectures on diffeomorphism groups in quantum physics ed J Govaerts, M N Hounkonnou
     and A Z Msezane *Contemporary Problems in Mathematical Physics: Proceedings of the Third International
     Conference, Cotonou, Benin* (Singapore: World Scientific) p 3
[32] Duff M J Kaluza-Klein theory in perspective *Preprint* hep-th/9410046
[33] Vilenkin N J 1968 *Group Representations* (Providence, RI: American Mathematical Society)
[34] Goldin G A and Sharp D H 1991 *Int. J. Mod. Phys.* B **5** 2625
[35] Goldin G A and Sharp D H 1996 *Phys. Rev. Lett.* **76** 1183
[36] Madore J 1999 *An Introduction to Noncommutative Differential Geometry and its Physical Applications*
     (Cambridge: Cambridge University Press)
[37] Lukierski J, Nowicki A, Ruegg H and Tolstoy V N 1991 *Phys. Lett.* B **264** 331
[38] Hochschild G and Serre J-P 1953 *Ann. Math.* **57** 591
[39] Kowalski-Glikman J and Smolin L 2004 *Phys. Rev.* D **70** 065020